SOLUTION OF MULTI-PARAMETER INVERSE PROBLEMS OF HEAT CONDUCTION

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We develop methods and present results for the solution of inverse problems of heat conduction in connection with the simultaneous determination of thermophysical characteristics and heat exchange parameters.

Introduction. In [1, 2] studies were made concerned with the possibility of an iterational filter for solving inverse problems in connection with the determination of a heat transfer parameter or a thermophysical characteristic (TPC); to do this, exterior or interior inverse problems of heat conduction (IPHC) were solved. Meanwhile, in the study of thermal processes in energetics, metallurgy, and other branches of technology, one frequently requires the simultaneous determination of several external and (or) internal parameters of thermal systems. Thus, for example, identification of a thermal flow $q(\tau)$ (heat transfer coefficient $\alpha(\tau)$) on cooling surfaces during the rolling of crude metals or the continuous casting of rods is not possible without reliable data on TPC, information concerning which can be of an extremely approximate nature. The necessity of solving combined IPHC in connection with the simultaneous determination of $\lambda(T)$ and $\alpha(\tau)(q(\tau))$ or $c_V(T)$ and $\alpha(\tau)(q(\tau))$ also arises when there is no reliable data on thermal flows (heat transfer coefficients and temperature T_c of the medium) for fully specifying boundary conditions for the solution of interior IPHC. It becomes necessary in this case to have knowledge of unknown heat transfer characteristics in the vector of identifying parameters.

Of most importance in the solution of IPHC are questions relating to identifiability of mathematical models. By identifiability we mean the possibility of determining the parameters of a model while preserving a one-to-one correspondence between them and the vector state. Studies made in [3] show that determination of the functions $\lambda(T)$ and $c_V(T)$ or $\lambda(T)$, $c_V(T)$, and $q(\tau)$ (or $\alpha(\tau)$) is possible when at least one of the given boundary conditions (BC) contains caloric quantities: $q(x, y, z, \tau)$ in a BC of the second kind, $\alpha(x, y, z, \tau)$ in a BC of the third kind or $\alpha(T)$ in a BC of the fourth kind (a necessary condition of identifiability). But if only zero BC of the second kind and (or) BC of the first kind are known, it is then necessary to specify at least one particular (frame of reference) value of the identifying functions.

Questions relating to uniqueness of a solution, closely connected with identifiability, will be discussed below after the introduction of studies of initial models of the IPHC being solved.

As a basic method we employed a pointwise identification of thermal parameters (individual values of the unknown functions are determined at each instant of time) without requiring a preliminary approximation of identifying parameters. We have demonstrated in [2, 4] the advantage of such an approach in the search for external and internal thermal parameters with use of an iterational filter.

With pointwise identification of BC and TPC, the unknown characteristics $q(\tau)$, $\alpha(\tau)$, and $T_c(\tau)$: are determined for each section of the heat exchange boundary, while the parameters $\lambda(T)$ and $C_V(T)$ are determined as functions of some mean-integral temperatures T_m, the method for choosing which was established in [2].

As an initial mathematical model for the solution of listed IPHC using a pointwise identification, we use the usual finite-difference approximations for the heat conduction equations and boundary conditions. A search of the width matrices appearing in computational procedures involving an algorithm of filtration requires modification of these equations amounting to a conversion of the unknown parameters into a state vector [4]. Examples of such conversions, used in the identification of $\lambda(T)$ and $c_V(T)$, are given in [2]. It becomes necessary here to write the

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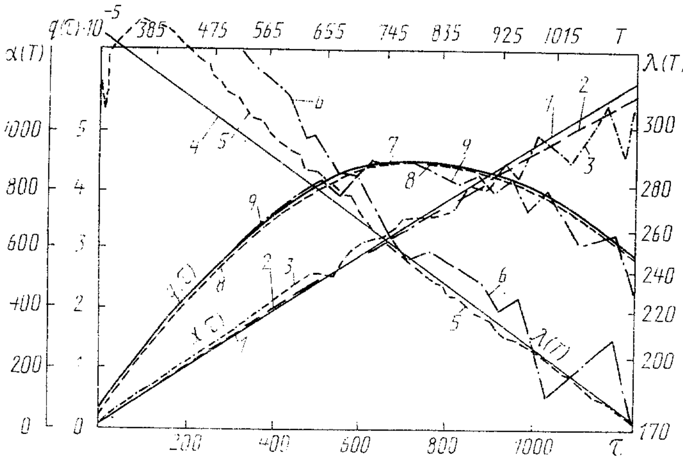

Fig. 1. Effect of measurement errors on the solution of a combined IPHC. α, W/(m^2·K); q, W/m^2; λ, W/(m·K), τ, sec, T °C.

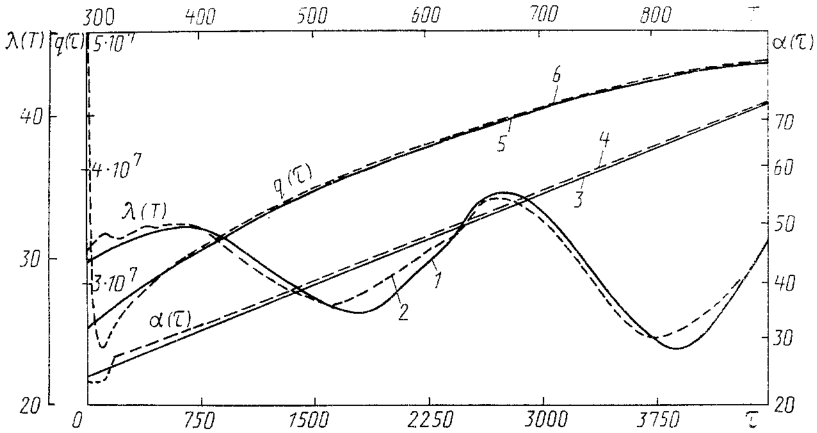

Fig. 2. Solution of combined IPHC with respect to identification. $\lambda(T)$, W/(m·K), $\alpha(\tau)$, W/(m^2·K).

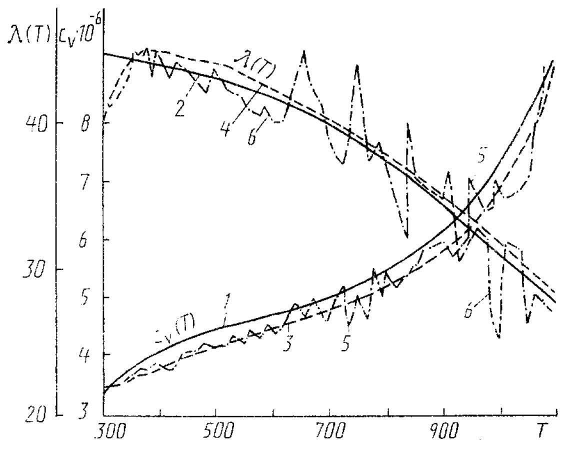

Fig. 3. Solution of internal IPHC. c_V, J/(m^2·K).

transformed finite-difference equations of heat conduction and BC of the second and third kinds for solution of multi-parameter internal and combined IPHC. Thus, in the identification of $\lambda(T)$ and $q(\tau)$ (or $\alpha(\tau)$), the indicated equations are written with $O(h^2 + \Delta\tau)$ precision in the following way (for simplicity we take the one-dimensional case with uniform grid, and the unknowns $\lambda(T_i)$, unlike those in [2], are replaced by $\lambda_m = \lambda(T_m)$:

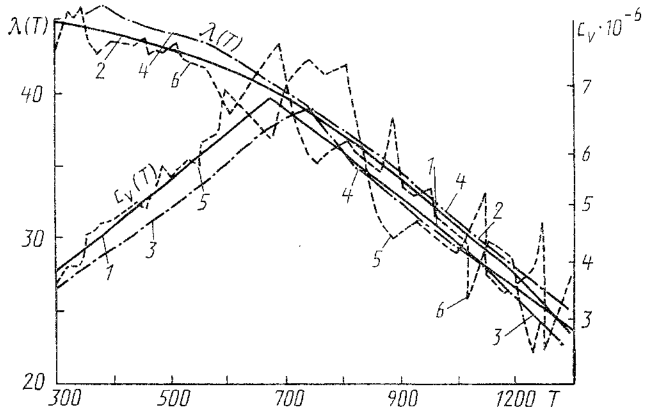

Fig. 4. Identification of $\lambda(T)$ and $c_V(T)$ for various values of mean-square error σ.

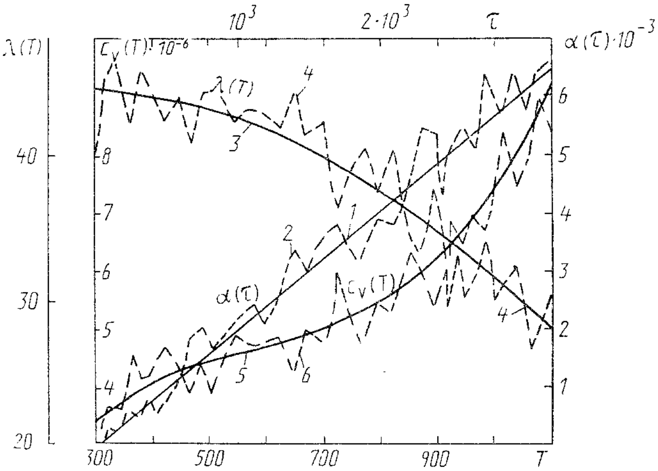

Fig. 5. Simultaneous identification of three thermal parameters $\lambda(T)$, $c_V(T)$, and $\alpha(\tau)$.

$$-\left[\frac{c_V(\hat{T}_i)^{(j/j-1)}_{k+1/k+1}}{\Delta\tau}\right](T_i)^{(j)}_{k+1}+\left[\frac{\hat{T}_{i+1}-2\hat{T}_i+\hat{T}_{i-1}}{h^2}\right]^{(j/j-1)}_{k+1/k+1}(\lambda_m)^{(j)}_{k+1}=$$
$$=-\left[\frac{c_V(\hat{T}_i)^{(j/j-1)}_{k+1}}{\Delta\tau}\right](T_i)^{(S)}_{k/k}; \quad (1)$$

$$\left[\frac{hc_V(\hat{T}_b)^{(j/j-1)}_{k+1/k+1}}{2\Delta\tau}\right](T_b)^{(j)}_{k+1}+\left[\frac{(\hat{T}_b-\hat{T}_{IN})^{(j/j-1)}_{k+1/k+1}}{h}\right](\lambda_m)^{(j)}_{k+1}-$$
$$-q^{(j)}_{k+1}=\left[\frac{hc_V(\hat{T}_b)^{(j/j-1)}_{k+1/k+1}}{2\Delta\tau}\right](T_b)^{(S)}_{k/k}; \quad (2)$$

$$\left[\frac{hc_V(\hat{T}_b)^{(j/j-1)}_{k+1/k+1}}{2\Delta\tau}\right](T_b)^{(j)}_{k+1}+\left[\frac{(\hat{T}_b-\hat{T}_{IN})^{(j/j-1)}_{k+1/k+1}}{h}\right](\lambda_b)^{(j)}_{k+1}+$$
$$+\left[(\hat{T}_b)^{(j/j-1)}_{k+1/k+1}-(T_c)_{k+1}\right]\alpha^{(j)}_{k+1}=\left[\frac{hc_V(\hat{T}_b)^{(j/j-1)}_{k+1/k+1}}{2\Delta\tau}\right](T_b)^{(S)}_{k/k}. \quad (3)$$

For simultaneous identification of $c_V(T)$ and $q(\tau)$ (or $\alpha(\tau)$) we obtain the following expressions (analogous to the preceding, $c_V(T_i)$ is replaced by $c_{Vm} = c_V(T_m)$):

$$\left[\frac{\lambda(\hat{T}_{i+1},\ \hat{T}_i)^{(j/j-1)}_{k+1/k+1}}{h^2}\right](T_{i+1})^{(j)}_{k+1}-\left\{\left[\frac{\lambda(\hat{T}_{i+1},\ \hat{T}_i)+\lambda(\hat{T}_{i-1},\ \hat{T}_i)}{h^2}\right]^{(j/j-1)}_{k+1/k+1}\right\}\times$$

$$\times (T_i)^{(j)}_{k+1} + \left[\frac{\lambda(\hat{T}_{i-1}, \hat{T}_i)^{(j/j-1)}_{k+1/k+1}}{h^2}\right](T_{i-1})^{(j)}_{k+1} - \left[\frac{(\hat{T}_i)^{(j/j-1)}_{k+1/k+1} - (\hat{T}_i)^{(S)}_{k/k}}{\Delta\tau}\right](c_{V_m})^{(j)}_{k+1} = 0; \quad (4)$$

$$\left[\frac{\lambda(\hat{T}_b, \hat{T}_{IN})^{(j/j-1)}_{k+1/k+1}}{h}\right](T_b)^{(j)}_{k+1} - \left[\frac{\lambda(\hat{T}_b, \hat{T}_{IN})^{(j/j-1)}_{k+1/k+1}}{h}\right](T_{IN})^{(j)}_{k+1} + \frac{h[(\hat{T}_b)^{(j/j-1)}_{k+1/k+1} - (\hat{T}_b)^{(S)}_{k/k}]}{2\Delta\tau}(c_{V_m})^{(j)}_{k+1} - q^{(j)}_{k+1} = 0; \quad (5)$$

$$\left[\frac{\lambda(\hat{T}_b, \hat{T}_{IN})^{(j/j-1)}_{k+1/k+1}}{h}\right](T_b)^{(j)}_{k+1} - \left[\frac{\lambda(\hat{T}_b, \hat{T}_{IN})^{(j/j-1)}_{k+1/k+1}}{h}\right] \times (T_{IN})^{(j)}_{k+1} + \frac{h[(\hat{T}_b)^{(j/j-1)}_{k+1/k+1} - (\hat{T}_b)^{(S)}_{k/k}]}{2\Delta\tau}(c_{V_m})^{(j)}_{k+1} + [(\hat{T}_b)^{(j/j-1)}_{k+1} - (T_c)_{k+1}]\,\alpha^{(j)}_{k+1} = 0. \quad (6)$$

To construct the coefficients in equations (1)-(6) at the $(k + 1)$-st time step we use predictors from the $(j - l)$-st to the jth iteration for the value of the temperature $(\hat{T}_i)_{k+1/k+1}^{(j/j-1)}$. In these equations T_b and T_{IN} are, respectively, temperatures on the boundary and at the closest interior node; h and $\Delta\tau$ are steps in the spatial and time coordinates; S is the number of iterations at the kth time step.

It is necessary to turn our attention separately to a simultaneous search for the thermal conductivity $\lambda(T)$ and the specific volumetric heat capacity $c_V(T)$. In this case the heat conduction equation, after transformations, will have the form

$$\left[\frac{(\hat{T}_{i+1} - 2\hat{T}_i + \hat{T}_{i-1})^{(j/j-1)}_{k+1/k+1}}{h^2}\right](\lambda_m)^{(j)}_{k+1} - \left[\frac{(\hat{T}_i)^{(j/j-1)}_{k+1/k+1} - (\hat{T}_i)^{(S)}_{k/k}}{\Delta\tau}\right](c_{V_m})^{(j)}_{k+1} = 0,$$

from which a decision can be made concerning degeneracy of the matrix formed from coefficients of the heat conduction equations and the boundary conditions. To eliminate this negative fact a step-by-step search is made for $\lambda(T)$ and $c_V(T)$. The essence of it is as follows. From some considerations, one specifies an initial approximation $(\hat{c}_{V_m})_{0/0}$, after which a search is made for $(\hat{\lambda}_m)_{1/1}$ in accordance with the method given in [2]. Following this, the quantity $(\hat{c}_{V_m})_{1/1}$ is determined similarly, having found $(\hat{\lambda}_m)_{1/1}$. This iterational process is repeated at each time step so long as the norms $\|(\hat{\lambda}_m)_{k/k}^{(l)} - (\hat{\lambda}_m)_{k/k}^{(l-1)}\|$ and $\|(\hat{c}_{V_m})_{k/k}^{(l)} - (\hat{c}_{V_m})_{k/k}^{(l-1)}\|$ are not less than some given ε_1 and ε_2 (here l and $l - 1$ are the numbers of two successive steps). This kind of method makes it possible to find $\gamma(T_m)$ and $c_V(T_m)$ with sufficient accuracy.

The advantages of the pointwise identification over the polynomial identification most frequently used becomes obvious if one considers at least some features of both types of identification. In the first place, accuracy in identifying the coefficients of a polynomial describing an unknown dependence depends essentially (when several characteristics are being determined simultaneously) on choice of the initial approximations for these coefficients (a large error in the choice of the initial estimates leads to oscillatory results). In pointwise identification the final solution is, in fact, independent of the initial approximation. Secondly, polynomial approximation does not allow one to carry out identification into the real time scale since sufficiently accurate estimates of the unknown coefficients can only be obtained at the end of the identification process, whereas, when characteristics are obtained pointwise, reliable values of the parameters may be determined at each instant of time. And, finally, in the third place, with polynomial identification there is unavoidable contradiction between accuracy of results and the stability of the solution of an ill-posed problem (as the degree of the polynomial increases, the accuracy gets better; however, from a certain moment on, a loss of stability is possible). With the pointwise approach, stability may be achieved by stopping the iterational process within each stage according to agreement of the total residual value with errors in the initial data, i.e., practically in accordance with the principle of the residual [5]. Of course, it is difficult to speak here about a complete agreement with the principle of the residual since for satisfaction of this principle we need, not a recurrent (stepwise) calculational procedure, but an algorithm operating over the whole time interval. However, in the problem being described (as in many of our earlier solutions [1-4]), instabilities in solutions were not observed when the indicated stoppage of the iterational process was used. In identifying more nonlinear relations in multi-dimensional formulations of IPHC, it is possible to increase stability with use of the step regularization procedure described here. Similar studies are being conducted by us at the present time.

Returning to the computational procedure for solving IPHC, we note that after completion of the solution at the next time step the diagonal elements of the covariance matrix $P_{k/k}$ are recovered to their original values (where, usually, the dispersions corresponding to $\lambda(T)$ and $c_V(T)$ are distinct). The non-diagonal elements of this matrix (correlation moments) are calculated from the usual expressions [1, 4]. As for passage from step to step within a time stage, we then employ covariant matrices here each time, these being obtained at the end of the last stage, depending on the determination, respectively, of $\lambda(T)$ or $c_V(T)$ at the preceding stage (or the initial matrices if this is the first stage).

The initial equations at each of the steps will have a different form. Thus, at the step of identification of $\lambda(T)$, the equations will have the form (1) from substitution into the coefficients the estimate $(\hat{c}_{V_m}{}^{\Pi})_{k+1/k+1}{}^{(j/j-1)}$, obtained at the preceding step. But at the step involving identification of $c_V(T)$, this equation will have the form (4) with coefficients depending on the quantities $(\hat{\lambda}_m{}^{\Pi})_{k+1/k+1}{}^{(j/j-1)}$, obtained at the step involving determination of $\lambda(T)$. Similar procedures are also carried out with the equations of the boundary conditions.

We now examine the possibilities of the method proposed for simultaneous determination of the three heat transport parameters: thermal conductivity $\lambda(T)$, specific volume heat capacity $c_V(T)$, and the heat transfer coefficient $\alpha(\tau)$ or the thermal flux $q(\tau)$.

Taking into account the approach presented above for simultaneous identification of λ and c_V, we seek the TPC stepwise, and the BD at an arbitrary one of the steps simultaneously with $\lambda(T)$. Here the initial equations will have the form (1)-(3) with corresponding substitution of the quantity $(\hat{c}_{V_m}{}^{\Pi})_{k+1/k+1}{}^{(j/j-1)}$ into the coefficients. Then at the second step of identifying $c_V(T)$ the heat conduction equation remains in the form (4), while the expression for the BC of the third kind can be written as follows:

$$\left[\frac{(\hat{\lambda}_m^{\Pi})_{k+1/k+1}^{(j/j-1)}}{h} + (\hat{\alpha}^{\Pi})_{k+1/k+1}^{(j/j-1)}\right](T_b)_{k+1}^{(j)} - \left[\frac{(\hat{\lambda}_m^{\Pi})_{k+1/k+1}^{(j/j-1)}}{h}\right](T_{IN})_{k+1}^{(j)} + \frac{h\,[(\hat{T}_b)_{k+1/k+1}^{(j/j-1)} - (\hat{T}_b)_{k/k}^{(S)}]}{2\Delta\tau}(c_{V_m})_{k+1}^{(j)} = (\hat{\alpha}^{\Pi})_{k+1/k+1}^{(j/j-1)}(T_c)_{k+1}. \tag{7}$$

Here $\hat{\alpha}^{\Pi}$ is the estimate for α obtained at the preceding stage of the $(k + 1)$-st time step.

The dimensions of the covariant matrix of errors of estimates of $P_{k/k}$ and, correspondingly, of all the other matrices appearing in the filtration algorithm, are different for different stages. Thus, if at the first stage the dimension of matrix $P_{k/k}$ is $n \times n$, then at the second stage it will be $(n - 1) \times (n - 1)$. As for variations in the number of elements of these matrices at the various stages, all that was said in this context, when the discussion concerned identification of $\lambda(T)$ and $c_V(T)$ remains valid.

It is necessary to say something concerning uniqueness of a solution of multi-parameter IPHC. A necessary condition for the possibility of simultaneous determination of several thermal parameters is the following. The number of unknowns in the space-time grid (discrete approximation of the heat conduction equation and boundary conditions) must not exceed the number of nodes in the measurements multiplied by the number of time steps. In other words, the general number of equations of measurements (in the space-time sense) must not be less than the number of parameters being identified.

Sufficient conditions for uniqueness of a solution of multi-parameter IPHC were given in [6-8].

Thus, for a thermal model with boundary conditions of the second and third kind [7]

$$\frac{\partial}{\partial x}\left\{\lambda(T)\frac{\partial T}{\partial x}\right\} = c_V(T)\frac{\partial T}{\partial \tau}; \tag{8}$$

$$\frac{\partial T(0,\ \tau)}{\partial x} = 0;\ \lambda(T)\frac{\partial T(1,\ \tau)}{\partial x} = q(\tau) \tag{9}$$

a solution of the problem of identifying $\lambda(T)$ and $c_V(T)$ is unique in the class of piecewise-analytic functions for temperature measurements on the boundaries.

For thermal measurements at internal points of a body the uniqueness of multi-parameter problems for identification of $\lambda(T)$, $Q(T)$, and $q(\tau)$ was studied in [8] ($Q(T)$ are internal heat sources). It was shown there that the parameters $\lambda(T)$ and $Q(T)$ are uniquely determined only in an interval between the minimum and maximum temperatures being measured even under a condition of monotonicity (piecewise monotonicity) of the temperature function. Outside of this interval the parameters being identified may not be determined uniquely. Apparently, however, if beforehand, or in the process of a numerical experiment, monotonicity or piecewise-monotonicity of the parameters being identified is found to be the case, we can then speak about uniqueness in their determination.

Returning to the model (4)-(7), we note that all the foregoing also applies to single-parameter IPHC considered here in connection with simultaneous determination of thermophysical characteristics or thermal conductivity and heat transfer coefficients.

It is necessary to dwell separately on the simultaneous search for λ, c_V, and α. The stepwise search for the parameters proposed here makes it possible to identify simultaneously only two parameters (α and λ), following which the third (c_B) can be identified. Therefore, the uniqueness conditions introduced here will also be valid. More to the point, the stepwise nature of the search for λ and c_V in connection with their pointwise identification does not permit posing the question concerning uniqueness of a simultaneous determination of TPC (only satisfaction of the first, necessary, condition of agreement of the number of measurement equations and the number of parameters being identified is required).

A solution of test problems allowed us to study possibilities of the proposed method and to estimate limits of regularity of the algorithms. As the object of investigation (combined IPHC were studied) we selected an infinite plate of thickness $L = 0.1$ m. "Measurements" were obtained by solving a direct problem with the following TPC:

$$\lambda(T) = 400 - 0{,}2T \ (\mathrm{W/m\cdot K}));$$

$$c_V(T) = \begin{cases} 1{,}5\cdot 10^6 + 8\cdot 10^3 T \ (\mathrm{J/(m^2\cdot K)}); & T < 683\,^\circ\mathrm{C}; \\ 1{,}2428\cdot 10^6 - 8\cdot 10^3 T \ (\mathrm{J/(m^2\cdot K)}); & T \geqslant 683\,^\circ\mathrm{C}. \end{cases}$$

On one of the surfaces homogeneous boundary conditions of the second kind, $\lambda(T)(\partial T/\partial x)|_{x=0} = 0$, were specified; on the other surface we specified a BC of the third kind: $\lambda(T)(\partial T/\partial x)|_{x=L} = \alpha(T - T_c)$; $\alpha(\tau) = 25 + \tau(\mathrm{W/(m\cdot K)})$; $T_c = 1400°C$. Steps of the space-time grid were: $h = 0.01$ m, $\Delta\tau = 20$ sec. In solving IPHC, to determine the effect on regularity of solutions of a measurement error we varied the latter from 0.4% to 3% of T_{max}. Initial estimates of the parameters $\lambda(T)$, $\alpha(\tau)$, and T, being identified, were selected fairly arbitrarily ($\hat{\lambda}_{0/0} = 320$; $\hat{\alpha}_{0/0} = 60$; $T_{0/0} = 280$). Results of the identification are shown in Fig. 1, where curves 1, 4, 7 (solid lines) correspond to nominal values of characteristics; curves 2, 5, 8 (dashed) correspond to estimates obtained for measurement errors of 0.4% of T_{max}; while curves 3, 6, 9 (dot-dash) are for $\sigma = 3\%$. Here curves 1, 2, 3 correspond to the heat transfer coefficient α; curves 4, 5, 6 to the thermal conductivity λ, and curves 7, 8, 9 to the thermal flux q. The maximum deviation of the solution from its nominal values does not exceed 8%. The coefficients $\alpha(\tau)$ which were determined made it possible to determine the thermal flow density. Naturally, the most qualitative results were obtained for measurements on the boundary where $\alpha(\tau)$ was determined, although no loss of stability was observed in other cases.

Studies were made in which $\lambda(T)$ was specified in the form of the sinusoid $\lambda(T) = T/200 \exp(T/200)\sin(\pi T/150) + 30$, with measurement error to within 5% of T_{max}. Here an increase of error did not result in a loss of stability. Actually, confirmation of this is shown by the results of identification of $\lambda(T)$ and $\lambda(\tau)$ in Fig. 2 (curves 1 and 3 are, respectively, nominal values of $\lambda(T)$ and $\alpha(\tau)$, while curves 2 and 4 are estimates of these same characteristics). The thermal flux density $q(\tau)$ was also obtained at the same time (curve 5 is the nominal value, curve 6 is an estimate).

We note that in the identification of several parameters an imprecision in obtaining one of them gives rise to a corresponding imprecision in obtaining the other. As for reconstructing the temperature field, the difference of the estimating values from the true values does not exceed 0.8%.

Simultaneous identification of $\lambda(T)$ and $c_V(T)$ was carried out for the same plate with heat transfer coefficients on the other boundary given by $\alpha(\tau) = 25 + 0.085\tau$.

To locate "measurements" we solved the direct problem with $\lambda(T) = 43.49 + 10.61\cdot 10^{-3}T - 23.02\cdot 10^{-6}T^2$; $c_V(T) = (-1.485 + 26.65\cdot 10^{-3}T - 41.19\cdot 10^{-6}T^2 + 23.42\cdot 10^{-9}T^3)\cdot 10^6$; and with initial distribution $T_0 = 293°C$. The time step was chosen to be equal to $\Delta\tau = 75$ sec. As "measurements" in solving IPHC, we took temperature at the nodes

4, 7, and 10, "decomposed" Gaussian white noise with σ varying from 0.1% to 3% of T_{max}. Initial approximations for the desired estimates were the following: $\hat{\lambda}_{0/0} = 40$; $\hat{c}_{V0/0} = 3.5\cdot10^6$, and $\mathbf{T}_{0/0} = 280$. Results of the identification are shown in Fig. 3, where curves 1, 2 (solid lines) are, respectively, specific volume heat capacity and thermal conductivity, having been used in solution of the direct problem; curves 3, 4 (dashed) are characteristics obtained for $\sigma = 0.1\%$ of T_{max}, while curves 5 and 6 (dash—dot) are for $\sigma = 3\%$ of T_{max}. As is evident from these graphs, an increase in σ from 0.1% to 3% results in an error increase in the estimates to 8% without a breakdown in stability. To investigate regularity of the identification process the problem considered above was solved for a specific volume heat capacity given in the form of the triangular impulse:

$$c_V(T) = \begin{cases} 1{,}5\cdot10^6 + 8\cdot10^3 T; & T < 683\,^\circ\mathrm{C}; \\ 12{,}428 - 8\cdot10^3 T; & T \geqslant 683\,^\circ\mathrm{C}. \end{cases}$$

All other parameters were left unchanged. Figure 4 shows the estimates corresponding to $\sigma = 0.1\%$ of T_{max} (dash-dot curves 3 and 4 are for $c_V(T)$ and $\lambda(T)$, respectively) and for $\sigma = 3\%$ of T_{max} (dashed curves 5 and 6 for $c_V(T)$ and $\lambda(T)$, respectively); it also shows the nominal values (continuous curves 1 and 2 for $c_V(T)$ and $\lambda(T)$, respectively). Here an increase in measurement errors from 0.1% to 3% results in a corresponding growth of errors in the estimates from 2% (curves 3 and 4) to 10% (curves 5 and 6) without loss of stability. In both cases the error in the temperature field did not exceed 1%. Thus, the form of the functions being identified has practically no effect on regularity of the solutions obtained with the aid of an iterational filter.

The most complicated IPHC is the problem of simultaneously identifying $\lambda(T)$, $c_V(T)$, and $\alpha(\tau)$ or $q(\tau)$. As the object of study we took the very same plate, but with boundary conditions of the third kind on both surfaces. To determine "measurements" we solved the direct problem with $\alpha = 25 + 2\tau$ and $T_c = 1400°C$ on one of the surfaces and with $\alpha = 200$ and $T_c = 293°C$ on the other. As TPC we took curves 1 and 2 (see Fig. 3). The space-time grid steps were: $h = 0.01$ m, $\Delta\tau = 10$ sec. As "measurable" quantities we took temperatures "distorted" by white noise ($\sigma = 3\%$) at nodes 2, 4, 7, 10. Initial estimates of the parameters being sought were: $\hat{\lambda}_{0/0} = 40$, $\hat{c}_{V0/0} = 3.5\cdot10^6$, $\alpha_{0/0} = 50$; $\mathbf{T}_{0/0} = 280$. In the first stage we identified $\lambda(T)$ and $\alpha(\tau)$. Here the covariant matrix of dimensions 13 × 13 had the form $P_{0/0}{}^{\lambda,\alpha}$ = diag {(1.6; 1.6; 1.6; 1.6; 1.6; 1.6; 1.6; 1.6; 1.6; 1.6; 1.6; 10^5; $3\cdot10^5$)$\cdot10^4$}. At the second stage (that of determining $c_V(T)$), in the covariant matrix $P_{0/0}{}^{c_V}$ of dimensions 12 × 12, the first 11 elements were the same, while the twelfth was chosen equal to $1.4\cdot10^4$.

Results obtained in solving this IPHC are shown in Fig. 5, where curves 1, 3, 5 are nominal characteristics of $\alpha(\tau)$, $\lambda(T)$, and $c_V(T)$, respectively, and curves 2, 4, 6 are estimates of these same characteristics corresponding to σ = 3% of T_{max}. Such an error in measurements leads to oscillations of the estimates about the nominal values with an amplitude of 25%. With measurement error decrease to 0.1%, the maximum amplitude of the oscillations decreases up to 5%.

Returning to the uniqueness of the solution of the problems described, we readily note that with an increase in temperatures the accuracy of estimates of TPC ($\lambda(T)$ in Fig. 1; $\lambda(T)$ and $c_V(T)$ in Fig. 3; $\lambda(T)$: in Fig. 4) becomes worse, and with an increase in time a similar situation (deterioration of the results) may be observed also for conditions of heat transfer ($\alpha(\tau)$ in Fig. 1). It is entirely possible that this is a consequence of a violation of the conditions of the uniqueness theorem in [8].

Conclusion. The algorithm and the solution results presented here for systematic problems testify to the effectiveness of the pointwise identification method presented for the simultaneous determination of several parameters of a thermal system and enable us to make a decision concerning the possibility of identifying some thermal parameters in real processes.

NOTATION

Here $\lambda(T)$ is thermal conductivity; T, temperature; $c_V(T)$, specific volume heat capacity; $\alpha(\tau)$, heat transfer coefficient; $q(\tau)$, thermal flux; x, y, z, τ space and time coordinates; j, S, number of iterations; k, number of time step; h and $\Delta\tau$, space and time steps; b and IN, subscripts indicating boundary and interior nodes; T_c, temperature of the medium; II, preceding step; σ, mean-square error; P, covariant matrix; ^, estimates of parameters and temperatures being identified; i, node number in spatial grid; T_m, mean integral temperature.

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NUMERICAL-ANALYTIC ALGORITHM OF THE STEFAN PROBLEM SOLUTION

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An algorithm is developed for the numerical solution of the Stefan problem for boundary conditions of the first, second, and third kinds, respectively, on the surface of a freezing (thawing) layer by using the solution of the heat conduction equation in the form of a series in the spatial coordinate including two derivatives of the time functions and their derivatives. An approximate estimation of the proposed method is given in an example of computing the freezing of water in a reservoir.

The necessity to determine the temperature state of objects being investigated with the natural-time change in the environment and the parameters governing it taken into account (the wind velocity, solar radiation, snow cover) occurs in solving many practical problems of engineering glaciology, geocryology, and metallurgy. Determination of the phase transition front of freezing water, soil, or a cooling metal ingot in a general formulation is a complex problem whose methods of solution still remain largely undeveloped [1-3]. This refers mainly to multidimensional problems with moving boundaries and one-dimensional problems with boundary conditions different from the first kind. Boundary conditions of the first kind for which the solution of the Stefan problem has been developed sufficiently completely assume the temperature of the surface of the freezing (thawing) massif to be given. In practice this yields results that are only qualitatively in agreement with the actual process. The algorithm considered below for the solution does not impose similar constraints and is similar in its content to the method of differential series [4, 5] but differs favorably from the latter in its clearness and simplicity. The formulated problem can be solved for any boundary conditions by an insignificant modification of one formula. Moreover, the temperature field of the freezing (thawing) or cooling layer needed for stress state computations and the motion law $\xi(\tau)$ of the phase interface boundary can be determined. The mathematical formulation of the problem has the form

$$\frac{\partial t_1(x,\ \tau)}{\partial\tau} = a_1\frac{\partial^2 t_1(x,\ \tau)}{\partial x^2},\ x\in[0,\ \xi(\tau)],\ \tau\in[0,\ \infty], \tag{1}$$

$$\frac{\partial t_2(x,\ \tau)}{\partial\tau} = a_2\frac{\partial^2 t_2(x,\ \tau)}{\partial x^2},\ x\in[\xi(\tau),\ \infty],\ \tau\in[0,\ \infty], \tag{2}$$

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